GEOMETRIC INTERSECTION NUMBER AND ANALOGUES OF THE CURVE COMPLEX FOR FREE GROUPS

ILYA KAPOVICH AND MARTIN LUSTIG

ABSTRACT. For the free group F_N of finite rank $N \geq 2$ we construct a canonical Bonahon-type, continuous and $Out(F_N)$-invariant *geometric intersection form*

$$\langle \ , \ \rangle : \overline{cv}(F_N) \times Curr(F_N) \to \mathbb{R}_{\geq 0}.$$

Here $\overline{cv}(F_N)$ is the closure of unprojectivized Culler-Vogtmann's Outer space $cv(F_N)$ in the equivariant Gromov-Hausdorff convergence topology (or, equivalently, in the length function topology). It is known that $\overline{cv}(F_N)$ consists of all *very small* minimal isometric actions of F_N on $\mathbb{R}$-trees. The projectivization of $\overline{cv}(F_N)$ provides a free group analogue of Thurston's compactification of the Teichmüller space.

As an application, using the *intersection graph* determined by the intersection form, we show that several natural analogues of the curve complex in the free group context have infinite diameter.

1. INTRODUCTION

The notion of an intersection number plays a crucial role in the study of Teichmüller space, mapping class groups, and their applications to 3-manifold topology. Thurston [54] extended the notion of a geometric intersection number between two free homotopy classes of closed curves on a surface to the notion of a *geometric intersection number* between two measured geodesic laminations. Indeed, this intersection number is a central concept in the study of Thurston's compactification of the Teichmüller space, as well as in the study of the dynamics and geometry of surface homeomorphisms. Bonahon [6] extended this notion of geometric intersection number to the case of two geodesic currents on the surface. Bonahon also constructed [7] a mapping class group equivariant embedding of Thurston's compactification of the Teichmüller space into the space of projectivized geodesic currents.

Culler and Vogtmann introduced in [19] a free-group analogue of Teichmüller space, which has been termed *Outer space* by Shalen and is denoted here by $CV(F_N)$ (where F_N is a free group of finite rank $N \geq 2$). Whereas points in Teichmüller space can be thought of as free and discrete

isometric actions of the surface group on $\mathbb{H}^2$, points in $CV(F_N)$ are represented by minimal free and discrete isometric actions of F_N on $\mathbb{R}$-trees with normalized co-volume (that is, where the quotient graph has volume 1). One also often works with the *unprojectivized Outer space* $cv(F_N)$, which contains a copy of $CV(F_N)$, and consists of all actions of the above type with arbitrary co-volume. More details are given in Section 2 below.

Let $\overline{cv}(F_N)$ be the closure of $cv(F_N)$ in the equivariant Gromov-Hausdorff topology. It is known [14, 5] that $\overline{cv}(F_N)$ consists precisely of all the minimal *very small* nontrivial isometric actions on F_N on $\mathbb{R}$-trees (see Section 2 for definitions). Projectivizing $\overline{cv}(F_N)$ gives a Thurston-type compactification $\overline{CV}(F_N) = CV(F_N) \cup \partial CV(F_N)$ of Outer Space, where $\partial CV(F_N)$ is the *Thurston boundary* of $CV(F_N)$. The outer automorphism group $Out(F_N)$ of F_N acts on $CV(F_N)$ and $\overline{CV}(F_N)$ in very close analogy to the action of the mapping class group on Teichmüller space and its Thurston compactification. One can regard $\partial cv(F_N) = \overline{cv}(F_N) - cv(F_N)$ as the *boundary* of $cv(F_N)$. The Thurston boundary $\partial CV(F_N)$ is obtained by projectivizing $\partial cv(F_N)$.

The structure of Outer space and of $Out(F_N)$ is more complicated than that of the Teichmüller space and the mapping class group. In large part this is due to the fact most free group automorphisms are not "geometric", in the sense that they are not induced by surface homeomorphisms. Although finite dimensional, $CV(F_N)$ is not a manifold, and hence none of the tools from complex analysis which are so useful for surfaces can be directly carried over into the free group world. Moreover, while the topological and homotopy properties of Outer space are fairly well understood, very little is known about the geometry of $CV(F_N)$. One of the reasons for this has been the lack, until recently, of a good geometric intersection theory in the Outer space context.

A *geodesic current* is a measure-theoretic generalization of the notion of a conjugacy class of a group element or of a free homotopy class of a closed curve on a surface (see Definition 5.4 below). Much of the motivation for studying currents comes from the work of Bonahon about geodesic currents on hyperbolic surfaces [6, 7]. The space $Curr(F_N)$ of all geodesic currents has a useful linear structure and admits a canonical $Out(F_N)$-action. The space $Curr(F_N)$ turns out to be a natural companion of the Outer space and contains additional valuable information about the geometry and dynamics of free group automorphisms. Examples of such applications can be found in [8, 18, 26, 33, 34, 35, 37, 42, 30, 48] and other sources.

In [34, 46] we introduced a Bonahon-type, continuous, and $Out(F_N)$-invariant *geometric intersection form*

$$\langle \ , \ \rangle : cv(F_N) \times Curr(F_N) \to \mathbb{R}_{\geq 0}.$$

The geometric intersection form $\langle \ , \ \rangle$ is $\mathbb{R}_{>0}$-homogeneous with respect to the first argument, $\mathbb{R}_{\geq 0}$-linear with respect to the second argument and is

$Out(F_N)$-equivariant. This intersection form has the following crucial property in common with Bonahon's notion of an intersection number between two geodesic currents on a surface:

For any $\mathbb{R}$-tree $T \in cv(F_N)$ and for any $g \in F_N \smallsetminus \{1\}$ we have $\langle T, \eta_g \rangle = ||g||_T$. Here η_g is the *counting current* of g (see Definition 5.6) and $||g||_T$ is the translation length of g on the $\mathbb{R}$-tree T. Since the scalar multiples of all counting currents form a dense set in $Curr(F_N)$, there is at most one continuous intersection form with the above properties, so that $\langle \ , \ \rangle$ is in fact canonical. Kapovich proved [34] that the intersection form $\langle \ , \ \rangle$ does not admit a "reasonable" continuous $Out(F_N)$-invariant symmetric extension to a map $Curr(F_N) \times Curr(F_N) \to \mathbb{R}$.

The main result of this paper is that the geometric intersection form $\langle \ , \ \rangle$ admits a continuous extension to the boundary of $cv(F)$. We present a simplified form of this result here and refer to Theorem 6.1 below for a more detailed statement.

Theorem A. *Let $N \geq 2$. There exists a unique continuous map*

$$\langle \ , \ \rangle : \overline{cv}(F_N) \times Curr(F_N) \to \mathbb{R}_{\geq 0}$$

which is $R_{\geq 0}$-homogeneous in the first argument, $R_{\geq 0}$-linear in the second argument, $Out(F_N)$-invariant, and such that for every $T \in \overline{cv}(F_N)$ and every $g \in F_N \smallsetminus \{1\}$ we have

$$\langle T, \eta_g \rangle = ||g||_T.$$

It is easy to see that the map $\langle \ , \ \rangle$ in Theorem A coincides with the intersection form from [34], when restricted to $cv(F_N) \times Curr(F_N)$.

Note that a very different and symmetric notion of an intersection number between two elements of $\overline{cv}(F_N)$ was introduced and studied by Guirardel [25]. However, Guiradel's intersection number often takes on the value ∞, and it is fairly difficult to use.

A key ingredient in the proof of Theorem A is Proposition 4.1 below, which establishes a "Uniform Scaling Approximation Property" for points in $\overline{cv}(F_N)$. It is clear that Proposition 4.1 should have further useful applications in the study of the boundary of the Outer space. The proof of Proposition 4.1 in turn relies on the *Bounded Back-Tracking Property* for very small actions of F_N on $\mathbb{R}$-trees, established by Gaboriau, Jaeger, Levitt, and Lustig in [20].

Recall that for a closed hyperbolic surface S the *curve graph* $\mathcal{C}(S)$ is defined as follows. The vertices of $\mathcal{C}(S)$ are free homotopy classes of essential simple closed curves on S. Two distinct vertices of $\mathcal{C}(S)$ are adjacent in $\mathcal{C}(S)$ if they can be realized by disjoint simple closed curves. The curve graph turned out to be a valuable tool in the study of the mapping class groups, of Kleinian groups, and in various applications to 3-manifolds.

Masur-Minsky [49] and Hempel [29] established that the curve graph has infinite diameter. Moreover, Masur-Minsky [49] proved that the curve graph is Gromov hyperbolic.

Algebraically, an essential simple closed curve α on S determines a splitting of $G = \pi_1(S)$ as an amalgamated free product or an HNN-extension over the cyclic subgroup generated by α (the amalgamated free product case occurs if α is separating and the HNN-extension case occurs if α is non-separating). Moreover, it is known [56] that all splittings of G over $\mathbb{Z}$ arise in this fashion.

In the free group context, it is often more natural to consider splittings over the trivial group rather than over $\mathbb{Z}$. Thus we define the *free splitting graph* $\mathcal{F} = \mathcal{F}(F_N)$ as follows. The vertices of $\mathcal{F}$ correspond to proper free product decompositions $F_N = A * B$, where $A \neq \{1\}, B \neq \{1\}$, where two such splittings are considered to be equal if their Bass-Serre trees are F_N-equivariantly isometric. Adjcency in $\mathcal{F}(F_N)$ corresponds to admitting a splitting of F_N that is a common refinement of the two splittings in question. Informally, two distinct splittings $F_N = A * B$ and $F_N = A' * B'$ are adjacent in $\mathcal{F}(F_N)$ if there exists a free product decomposition $F_N = C_1 * C_2 * C_3$ such that either $A = C_1 * C_2$, $B = C_3$ and $A' = C_1$, $B' = C_2 * C_3$, or else $A' = C_1 * C_2$, $B' = C_3$ and $A = C_1$, $B = C_2 * C_3$. It is not hard to see that $\mathcal{F}_N$ is connected for $N \geq 3$. We also define the *dual free splitting graph* $\mathcal{F}^* = \mathcal{F}^*(F_N)$ as follows. The vertex set of $\mathcal{F}^*(F_N)$ is the same as the vertex set of $\mathcal{F}(F_N)$. Two vertices T_1 and T_2 of $\mathcal{F}^*(F_N)$ are adjacent in $\mathcal{F}^*(F_N)$ if there exists a nontrivial element $a \in F_N$ which is elliptic with respect to both T and T_1, that is $||a||_{T_1} = ||a||_{T_2}$ (this adjacency condition turns out be be equivalent to requiring that there exist a primitive, i.e. a member of a free basis, element of F_N that is elliptic for both T_1 and T_2). In the standard curve complex context, analogues of definitions of adjacency in $\mathcal{F}$ and $\mathcal{F}^*$ are essentially equivalent to the standard definition of adjacency in $\mathcal{C}(S)$. Namely, two non-isotopic simple closed curves define adjacent vertices of $\mathcal{C}(S)$ if and only if the corresponding cyclic splittings of $\pi_1(S)$ have a common refinement. Also, two such curves define vertices at distance ≤ 2 in $\mathcal{C}(S)$ if and only if the corresponding splittings of $\pi_1(S)$ have a common nontrivial elliptic element. However, in the context of free groups $\mathcal{F}$ and $\mathcal{F}^*$ appear to be rather different objects, with distances in $\mathcal{F}^*$ often being much smaller than in $\mathcal{F}$.

We also introduce a key new object $\mathcal{I}(F_N)$ called the *intersection graph* of F_N. The graph $\mathcal{I}(F_N)$ is a bipartite graph with vertices of two kinds: projective classes $[T]$ of very small F_N-trees $T \in \overline{cv}(F_N)$ and projective classes $[\mu]$ of nonzero currents $\mu \in Curr(F_N)$. Two vertices $[T]$ and $[\mu]$ are adjacent in $\mathcal{I}(F_N)$ whenever $\langle T, \mu \rangle = 0$. For $N \geq 3$ the graph $\mathcal{I}(F_N)$ has a large $Out(F_N)$-invariant connected component $\mathcal{I}_0(F_N)$ that contains all projective classes of Bass-Serre trees T corresponding to nontrivial free

product decompositions of F_N (this component also contains all the projective currents $[\eta_a]$ corresponding to primitive elements a of F_N). Both $\mathcal{F}(F_N)$ and $\mathcal{F}^*(F_N)$ admit $Out(F_N)$-equivariant Lipshitz maps into $\mathcal{I}_0(F_N)$ as do essentially all other reasonable analogues of the notion of a curve complex for free groups.

The results proved in sections 7 and 8 of this paper can be summarized as follows:

Theorem B. *Let $N \geq 3$. Then the graphs $\mathcal{I}_0(F_N)$, $\mathcal{F}(F_N)$ and $\mathcal{F}^*(F_N)$ have infinite diameter.*

Moreover, if Y_N is one of the above graphs and $\phi \in Out(F_N)$ is an atoroidal iwip, i.e. ϕ is irreducible with irreducible powers (see Definition 7.3) and has no periodic conjugacy classes in F_N, then for any vertices x, y of Y_N we have

$$\lim_{n \to \infty} d_{Y_N}(x, \phi^n y) = \infty.$$

Recently Behrstock, Bestvina and Clay [2] obtained by different arguments an independent proof the conclusion of Theorem B for the complex $\mathcal{S}(F_N)$ which is quasi-isometric to $\mathcal{F}(F_N)$ (see Definition 8.4 and the subsequent discussion).

Our proof of Theorem B also shows that the "directions to infinity", given by Theorem B, corresponding to substantially different ϕ, are distinct. Thus one can also show that if ψ and ϕ are two elements of $Out(F_N)$ such that they are irreducible with irreducible powers and without periodic conjugacy classes, and such that the subgroup $\langle \phi, \psi \rangle$ is not virtually cyclic then for any $x \in VY_N$ and any sequences $n_i \to \infty$, $m_i \to \infty$ we have

$$\lim_{i \to \infty} d_{Y_N}(\phi^{n_i} x, \psi^{m_i} x) = \infty.$$

We also consider several natural variations of $\mathcal{F}(F_N)$ and $\mathcal{F}^*(F_N)$ and note that conclusions of Theorem B apply to them as well. Note that the intersection graph $\mathcal{I}(F_N)$ is not connected and it has other interesting connected components apart from $I_0(F_N)$. For example, for any $T \in cv(F_N)$ and for any current μ with full support, $[T]$ and $[\mu]$ are isolated vertices of $\mathcal{I}(F_N)$. Similarly, if $\phi \in Out(F_N)$ is an atoroidal iwip then for the "stable eigentree" $T(\phi)$ and "stable eigencurrent" $\mu(\phi)$ the pair $[T(\phi)], [\mu(\phi)]$ forms an isolated edge in $\mathcal{I}(F_N)$ (see [38] for details).

It seems plausible that the graphs $\mathcal{F}(F_N)$ and $\mathcal{F}^*(F_N)$ are not quasi-isometric. Investigating hyperbolicity properties of these graphs remains an interesting open problem for future study. It appears that $\mathcal{F}(F_N)$ is unlikely to be Gromov-hyperbolic while $\mathcal{F}^*(F_N)$ does seem to have a chance for hyperbolicity. In particular, a reducible element of $Out(F_N)$ always acts on $\mathcal{F}^*(F_N)$ with a bounded orbit while it seems possible for a reducible automorphism to act on $\mathcal{F}(F_N)$ with an unbounded orbit. Also, it seems that the orbits in $\mathcal{F}(F_N)$ of free abelian subgroups of $Out(F_N)$ may produce quasi-flats, while similar orbits in $\mathcal{F}^*(F_N)$ have finite diameter.

The main result of this paper, Theorem A, seems to have the potential to become a important tool in the study of Outer space and $Out(F_N)$:

- We prove in sections 7 and 8 of this paper that several free group analogues of the curve complex have infinite diameter.
- Ursula Hamenstädt had recently used [27] Theorem A as a crucial ingredient to prove that any non-elementary subgroup of $Out(F_N)$, where $N \geq 3$, has infinite dimensional second bounded cohomology group (infinite dimensional space of quasi-morphisms). This in turn has an application to proving that any homomorphism from any lattice in a higher-rank semi-simple Lie group to $Out(F_N)$ has finite image.

- Very recently Bestvina and Feighn [10] used Theorem A as a key tool in the proof that for any finite collection $\phi_1, \ldots, \phi_m \in Out(F_N)$ of iwip (irreducible with irreducible powers, see Definition 7.3) outer automorphisms of F_N there exists a δ-hyperbolic complex X with an isometric $Out(F_N)$-action where each ϕ_i acts with a positive translation length. Unlike the curve complex analogues discussed here in section 7 and 8, the Bestvina-Feighn construction is not intrinsically defined, but their result gives substantial hope and significant indication that some of these other more functorial and intrinsic analogues of the curve complex for free groups may be Gromov-hyperbolic as well.
- The results of the new paper [10] of Bestvina and Feighn also imply that if $\phi \in Out(F_N)$ is an iwip, then ϕ acts with positive asymptotic translation length on $\mathcal{F}(F_N)$ and on $\mathcal{F}^*(F_N)$. This means that when Y_N is one of these two graphs, and $T \in VY_N$ is an arbitrary vertex, then the orbit map $\mathbb{Z} \to Y_N, n \mapsto \phi^n T$, is a quasi-isometric embedding.
- In a new preprint [39], we use Theorem A to construct domains of discontinuity for the action of subgroups of $Out(F_N)$ on $\overline{CV}(F_N)$ and on $\mathbb{P}Curr(F_N)$.
- In another new preprint [40] we show that every subgroup of $Out(F_N)$ which contains an atoroidal iwip and is not virtually cyclic, also contains a non-abelian free subgroup where every non-trivial element is an atoroidal iwip.

- Finally, in [38] we use Theorem A to characterize the situation where $\langle T, \mu \rangle = 0$. Specifically, we prove in [38] that for $T \in \overline{cv}(F_N)$ and $\mu \in Curr(F_N)$ we have $\langle T, \mu \rangle = 0$ if and only if $supp(\mu) \subseteq L^2(T)$. Here $supp(\mu)$ is the support of μ and $L^2(T)$ is the *dual algebraic lamination* of T (see [16]). That result in turn is applied in [38] to the notions of a *filling conjugacy class* and a *filling current* as well as to obtain results about *bounded translation equivalence* in F_N. In [38] we also obtain a generalization of the length compactness result of Francaviglia [26]: we show that if $T \in cv(F_N)$ and $\mu \in Curr(F_N)$ is

a current with full support (e.g. the Patterson-Sullivan current [42]) then for every $C > 0$ the set $\{\phi \in Out(F_N) : \langle T, \phi\mu \rangle \leq C\}$ is finite and hence the set $\{\langle T, \phi\mu \rangle : \phi \in Out(F_N)\} \subseteq \mathbb{R}$ is discrete.

2. Outer space and its closure

We will only briefly recall the main definitions related to Outer space here. For a more detailed background information we refer the reader to [3, 14, 19, 24, 47] and other sources.

Let F_N be a free group of finite rank $N \geq 2$. Let T be an $\mathbb{R}$-tree with an isometric action of F_N. For any $g \in F_N$, denote

$$||g||_T = \inf_{x \in T} d_T(x, gx) = \min_{x \in T} d_T(x, gx).$$

The number $||g||_T$ is called the *translation length* of g.

Remark 2.1. Note that for all $m \in \mathbb{Z}$ we have:

$$||g^m||_T = |m| \cdot ||g||_T.$$

An isometric action of F_N on an $\mathbb{R}$-tree T is called *minimal* if T has no proper F_N-invariant subtrees.

Definition 2.2. An isometric action of F_N on an $\mathbb{R}$-tree T action is called *very small* if:

(1) The stabilizer in F_N of every non-degenerate arc in T is cyclic (either trivial or infinite cyclic).
(2) The stabilizer in F_N of every non-degenerate tripod is trivial.
(3) For every $g \in F_N, g \neq 1$ and every integer $n \neq 0$ if g^n fixes some non-degenerate arc, then g fixes that arc.

Thus free isometric actions of F_N on $\mathbb{R}$-trees, and, more generally, actions with trivial arc stabilizers, are very small.

Definition 2.3 (Outer space and its closure). Let $N \geq 2$ be an integer.

(1) We denote by $cv(F_N)$ the space of all minimal free and discrete isometric actions of F_N on $\mathbb{R}$-trees. Two such actions of F_N on trees T and T' are identified in $cv(F_N)$ if there exists an F_N-equivariant

isometry between T and T'. The space $cv(F_N)$ is called *unprojectivized Outer space* for F_N.

(2) Denote by $\overline{cv}(F_N)$ the space of all minimal nontrivial very small isometric actions of F_N on $\mathbb{R}$-trees. Again, two such actions are considered equal in $\overline{cv}(F_N)$ if there exists an F_N-equivariant isometry between the two trees in question.

Note that if $T \in cv(F_N)$ then the quotient T/F_N is compact. It is known that every element $T \in \overline{cv}(F_N)$ is uniquely identified by its *translation length function* $F_N \to \mathbb{R}$, $g \mapsto ||g||_T$. That is, for $T, T' \in \overline{cv}(F_N)$ we have $T = T'$ if and only if $||g||_T = ||g||_{T'}$ for all $g \in F_n$.

The spaces $cv(F_N)$ and $\overline{cv}(F_N)$ have several natural topologies that are all known to coincide [50]: the pointwise translation length function convergence topology, the equivariant Gromov-Hausdorff-Paulin convergence topology and the weak CW-topology (for the case of $cv(F_N)$). In particular if $T_n, T \in \overline{cv}(F_N)$ then $\lim_{n \to \infty} T_n = T$ if and only if for every $g \in F_N$ we have $\lim_{n \to \infty} ||g||_{T_n} = ||g||_T$. Note that $cv(F_N) \subseteq \overline{cv}(F_N)$. It is known that $\overline{cv}(F_N)$ is precisely the closure of $cv(F_N)$ (with respect to either of the above topologies).

There is a natural *continuous action* of $Out(F_N)$ on $\overline{cv}(F_N)$ that preserves $cv(F_N)$, and which can be written from the left as well as from the right, using the convention $\phi T = T\phi^{-1}$ for $T \in \overline{cv}(F_N)$ and $\phi \in Out(F_N)$. At the translation-length-function level this action can be defined as follows. For $T \in \overline{cv}(F_N)$ and $\widehat{\varphi} \in Aut(F_N)$ with image $\varphi \in Out(F_N)$ we have

$$||g||_{T\varphi} = ||g||_{\varphi^{-1}T} = ||\widehat{\varphi}(g)||_T$$

for any $g \in F_N$.

Definition 2.4 (Projectivized Outer space and its compactification).
(1) For $N \geq 2$ one defines $CV(F_N) = cv(F_N)/\sim$, where $T_1 \sim T_2$ whenever there exists an F_N-equivariant homothety between T_1 and T_2. Thus $T_1 \sim T_2$ in $cv(F_N)$ if and only if there is a constant $c > 0$ such that $||g||_{T_1} = c \cdot ||g||_{T_2}$ for all $g \in F_N$. The space $CV(F_N)$, first introduced by M. Culler and K. Vogtmann [19], is called the *projectivized Outer space* or simply *Outer space*.

(2) Similarly, define $\overline{CV}(F_N) = \overline{cv}(F_N)/\sim$ where $\sim$ is again the above homothety relation. For $T \in \overline{cv}(F_N)$ denote by $[T]$ the $\sim$-equivalence class of T.

(c) The spaces $CV(F_N)$ and $\overline{CV}(F_N)$ inherit the quotient topology from $cv(F_N)$ and $\overline{cv}(F_N)$. This makes the inclusion $CV(F_N) \subseteq \overline{CV}(F_N)$ into a topological embedding with dense image. Moreover, the space $\overline{CV}(F_N)$ is compact and thus provides a natural compactification of $CV(F_N)$. We also denote $\partial CV(F_N) = \overline{CV}(F_N) \smallsetminus CV(F_N)$ and call $\partial CV(F_N)$ the *Thurston boundary* of $CV(F_N)$.

The natural action of $Out(F_N)$ on $\overline{cv}(F_N)$ factors through to the action of $Out(F_N)$ by homeomorphisms on $\overline{CV}(F_N)$. Namely, for $\varphi \in Out(F_N)$ and $T \in \overline{cv}(F_N)$ we have $\varphi[T] = [\varphi T]$. This action of $Out(F_N)$ on $\overline{CV}(F_N)$ leaves $CV(F_N)$ invariant, so that $Out(F_N)$ acts on $CV(F_N)$ as well.

Remark 2.5. There is a standard $Out(F_N)$-equivariant topological embedding $j : CV(F_N) \to cv(F_N)$ that gives the identity on $CV(F_N)$ when composed with the projection map $cv(F_N) \to cv(F_N)/\sim = CV(F_N)$. Namely, $j([T]) = T'$, where $T' \sim T$ and the quotient graph T'/F_N has volume 1. One can alternatively think about elements of $cv(F_N)$ as *marked metric graph structures* on F_N, as explained in more detail in Remark 5.2 below.

3. Bounded back-tracking

As before let F_N be a free group of finite rank $N \geq 2$, and let A be a free basis of F_N. We denote by T_A the Cayley graph (which, of course, is a tree !) of F_N with respect to A.

Let T be an $\mathbb{R}$-tree with an isometric action of F_N, and consider a point $p \in T$. There is a unique F_N-equivariant map $i_p : T_A \to T$ which is linear on each edge of T_A, and which satisfies $i_p(1) = p$.

Definition 3.1 (Bounded Back-Tracking constant)**.** The *bounded back-tracking constant* with respect to A, T and p, denoted $BBT_{p,A}(T)$, is the infimum of all constants $C > 0$ such that for any $Q, R \in T_A$, the image $i_p([Q, R])$ of $[Q, R] \subseteq T_A$ is contained in the C-neighborhood of $[i_p(Q), i_p(R)]$.

An useful result of [20] states:

Proposition 3.2. *Let F_N be a finitely generated non-abelian free group with a minimal very small isometric action on an $\mathbb{R}$-tree T. Let A be a free basis of F_N and let $p \in T$.*
Then we have:
$$BBT_{p,A}(T) \leq \sum_{a \in A} d_T(p, ap).$$
In particular, $BBT_{p,A}(T) < \infty$.

The following is an easy corollary of the definitions (see Lemma 3.1(b) of [20] or Lemma 3.1 of [17]):

Lemma 3.3. *Let F_N be a finitely generated non-abelian free group with a minimal very small isometric action on an $\mathbb{R}$-tree T. Let A be a free basis of F_N and let $p \in T$.*
Suppose $BBT_{p,A}(T) < C$. Then the following hold:

(1) *Let $w \in F(A)$ be cyclically reduced. Then*
$$\left| \|w\|_T - d_T(p, wp) \right| \leq 2C.$$

(2) *Let $u = u_1 \ldots u_m$ be a freely reduced product of freely reduced words in $F = F(A)$, where $m \geq 1$. Then we have*

$$\left| d_T(p, up) - \sum_{i=1}^{m} d_T(p, u_i p) \right| \leq 2mC.$$

(3) *Suppose $u, u_1, \ldots, u_m$ are as in (2) and that, in addition, u is cyclically reduced in $F(A)$. Then*

$$\left| \|u\|_T - \sum_{i=1}^{m} d_T(p, u_i p) \right| \leq 2mC + 2C \leq 4mC.$$

(4) *Suppose $u, u_1, \ldots, u_m$ are as in (2) and that, in addition, $u, u_1, \ldots, u_m$ are cyclically reduced in $F(A)$. Then*

$$\left| \|u\|_T - \sum_{i=1}^{m} \|u_i\|_T \right| \leq 6mC.$$

4. Uniform approximation of $\mathbb{R}$-trees

Let A be a free basis of F_N. Recall that T_A is the Cayley tree of F_N with respect to A, where all edges of T_A have length 1. Thus $T_A \in cv(F_N)$. For $g \in F_N$ we denote by $|g|_A$ the freely reduced length of g with respect to A, and we denote by $\|g\|_A$ the cyclically reduced length of g with respect to A. Thus $\|g\|_A = \|g\|_{T_A}$.

The following statement is a key ingredient in the proof of the continuity of our geometric intersection number. We believe that it will also turn out to be useful in other circumstances.

Proposition 4.1 (Uniform Scaling Approximation). *Let $T \in \overline{cv}(F_N)$, let A be a free basis of F_N and let $\epsilon > 0$. Then there exists a neighborhood U_ϵ of T in $\overline{cv}(F_N)$, such that for every $w \in F_N$ and every $T_1, T_2 \in U_\epsilon$ we have:*

(†) $$\left| \|w\|_{T_1} - \|w\|_{T_2} \right| \leq \epsilon \|w\|_A.$$

Proof. Choose $p \in T$. Let $C > 0$ be such that $d_T(p, ap) < C/N$ for every $a \in A$, so that by Proposition 3.2 we have $BBT_{p,A}(T) < C$. It suffices to prove the proposition for all sufficiently small ϵ, and we will assume that $\epsilon > 0$ satisfies $N\epsilon \leq C$.

Choose an integer $M > 1$ so that $16C/M < \epsilon/2$. Let $0 < \epsilon_1 < \epsilon$ be such that $\frac{2\epsilon_1}{M} \leq \epsilon/2$.

Using the equivariant Gromov-Hausdorff-Paulin topology on $\overline{cv}(F_N)$ it follows that there exists a neighborhood U_ϵ of T in $\overline{cv}(F_N)$ such that for every $T' \in U_\epsilon$ the following holds: There is some $p' \in T'$ such that for every $g \in F_N$ with $|g|_A \leq M$ we have

(∗) $$\left| d_T(p, gp) - d_{T'}(p', gp') \right| \leq \epsilon_1.$$

Hence $BBT_{p',A}(T') \leq \sum_{a \in A} d_{T'}(p', ap') < C + N\epsilon_1 \leq 2C$. We will now verify that the neighborhood U_ϵ satisfies the requirements of the proposition.

Let $T_1, T_2 \in U_\epsilon$ be arbitrary, and let $p_1 \in T_1, p_2 \in T_2$ be chosen as above. Let $w \in F(A)$ be a non-trivial cyclically reduced word such that $||w||_A$ is divisible by M. Put $m = ||w||_A/M$. Thus $m \geq 1$ is an integer. Write w as a freely reduced product $w = u_1 \ldots u_m$ in $F(A)$, where $|u_i|_A = M$ for all $i = 1, \ldots, m$.

Then, by the properties of the BBT-constant listed in Lemma 3.3 (specifically, by part (3) of Lemma 3.3), we have for $j = 1, 2$:

$$\left| ||w||_{T_j} - \sum_{i=1}^{m} d_{T_j}(p_j, u_i p_j) \right| \leq 8Cm$$

Also, $(*)$ implies that for $j = 1, 2$ the inequality

$$\left| \sum_{i=1}^{m} d_T(p, u_i p) - \sum_{i=1}^{m} d_{T_j}(p_j, u_i p_j) \right| \leq m\epsilon_1$$

holds. This implies:

$$\left| ||w||_{T_1} - ||w||_{T_2} \right| \leq 16Cm + 2m\epsilon_1 = \frac{16C + 2\epsilon_1}{M} ||w||_A \leq \epsilon ||w||_A$$

Thus we have established that $(\dagger)$ holds for every $w \in F_N$ with $||w||_A$ divisible by M.

For the general case let $w \in F(A)$ be an arbitrary nontrivial cyclically reduced word. Since $||w^M||_A = M||w||_A$ is divisible by M, we already know that $(\dagger)$ holds for w^M. By dividing by M both sides of the inequality $(\dagger)$ for w^M, we conclude that $(\dagger)$ holds for w in view of Remark 2.1.   □

5. Geodesic currents

Let ∂F_N be the hyperbolic boundary of F_N (see [22] for background information about word-hyperbolic groups). We denote

$$\partial^2 F_N = \{(\xi_1, \xi_2) : \xi_1, \xi_2 \in \partial F_N, \text{ and } \xi_1 \neq \xi_2\}.$$

Also denote by $\sigma : \partial^2 F_N \to \partial^2 F_N$ the "flip" map defined as $\sigma : (\xi_1, \xi_2) \mapsto (\xi_2, \xi_1)$ for $(\xi_1, \xi_2) \in \partial^2 F_N$.

Definition 5.1 (Simplicial charts). A *simplicial chart* on F_N is an isomorphism $\alpha : F_N \to \pi_1(\Gamma, x)$, where Γ is a finite connected graph without valence-one vertices, and where x is a vertex of Γ.

A simplicial chart α on F_N defines an F_N-equivariant quasi-isometry between F_N (with any word metric) and the universal covering $\widetilde{\Gamma}$, equipped with the simplicial metric (i.e. every edge has length 1). Correspondingly, we get canonical F_N-equivariant homeomorphisms $\partial\alpha : \partial F_N \to \partial\widetilde{\Gamma}$ and $\partial^2\alpha : \partial^2 F_N \to \partial^2\widetilde{\Gamma}$, that do not depend on the choice of a word metric for F_N. If α is fixed, we will usually use these homeomorphisms to identify ∂F_N with $\partial\widetilde{\Gamma}$ and $\partial^2 F_N$ with $\partial^2\widetilde{\Gamma}$ without additional comment.

Remark 5.2.

(a) Combinatorially, we adopt Serre's convention regarding graphs. Thus every edge $e \in E\Gamma$ comes equipped with the *inverse edge* e^{-1}, such that $e \neq e^{-1}$ and $(e^{-1})^{-1} = e$. Moreover, for every $e \in E\Gamma$, the initial vertex of e is the terminal vertex of e^{-1} and the terminal vertex of e is the initial vertex of e^{-1}. An *orientation* on Γ is a partition $E\Gamma = E^+\Gamma \sqcup E^-\Gamma$ such that for every $e \in E\Gamma$ one of the edges e, e^{-1} belongs to $E^+\Gamma$ and the other belongs to $E^-\Gamma$.

(b) Any simplicial chart $\alpha : F_N \to \pi_1(\Gamma, x)$ defines a finite-dimensional open cell in $cv(F_N)$ and a finite-dimensional open simplex in $CV(F_N)$. More precisely, let L be a *metric graph structure* on Γ, that is, a map $L : E\Gamma \to R_{>0}$ such that $L(e) = L(e^{-1})$ for every edge $e \in E\Gamma$. Then we can pull-back L to $\widetilde{\Gamma}$ by giving every edge in $\widetilde{\Gamma}$ the same length as that of its projection in Γ. Let d_L be the resulting metric on $\widetilde{\Gamma}$, which makes $\widetilde{\Gamma}$ into an $\mathbb{R}$-tree. The action of F_N on this tree, defined via α, is a deck transformation action and thus minimal, free and discrete. Hence this action defines a point in $cv(F_N)$. Varying the metric structure L on Γ produces an open cone $\Delta(\alpha) \subseteq cv(F_N)$ in $cv(F_N)$, which is homeomorphic to the positive open cone in $\mathbb{R}^m$. Here m is the number of topological edges of Γ, that is, $m = \frac{1}{2}\#E\Gamma$. Thus we can think of a simplicial chart $\alpha : F_N \to \pi_1(\Gamma, x)$ as defining a local "coordinate patch" on $cv(F_N)$.

(c) If we require the sum of the lengths of all the topological edges of Γ to be equal to 1, we get a subset $\Delta'(\alpha)$ of $cv(F_N)$ that is homeomorphic to an open simplex of dimension $m - 1$. This subset $\Delta'(\alpha)$ belongs to the subset $j(CV(F_N))$ defined in Remark 2.5, and hence projects homeomorphically to its image in $CV(F_N)$.

(d) Moreover, the union of open cones $\Delta(\alpha)$ over all simplicial charts α is equal to $cv(F_N)$, and this union is a disjoint union. Additionally, every point of $cv(F_N)$ belongs to only a finite number of closures $\overline{\Delta}(\alpha)$ of such open cones. Similarly, the copy $j(CV(F_N))$ of $CV(F_N)$ in $cv(F_N)$ is the disjoint union of the open simplices $\Delta'(\alpha)$ over all simplicial charts α, and the closures of these open simplices in $cv(F_N)$ form a locally finite cover of $j(CV(F_N))$.

Definition 5.3 (Cylinders)**.** Let $\alpha : F_N \to \pi_1(\Gamma, x)$ be a simplicial chart on F_N. For a non-trivial reduced edge-path γ in $\widetilde{\Gamma}$ denote by $Cyl_{\widetilde{\Gamma}}(\gamma)$ the set of all $(\xi_1, \xi_2) \in \partial^2 F_N$ such that the bi-infinite geodesic from $\tilde{\alpha}(\xi_1)$ to $\tilde{\alpha}(\xi_2)$ contains γ as a subpath.

We call $Cyl_{\widetilde{\Gamma}}(\gamma) \subseteq \partial^2 F_N$ the *two-sided cylinder corresponding to* γ.

It is easy to see that $Cyl_{\widetilde{\Gamma}}(\gamma) \subseteq \partial^2 F_N$ is both compact and open. Moreover, the collection of all such cylinders, where γ varies over all non-trivial reduced edge-paths in $\widetilde{\Gamma}$, forms a basis of open sets in $\partial^2 F_N$.

Definition 5.4 (Geodesic currents)**.** A *geodesic current* (or simply *current*) on F_N is a positive Radon measure on $\partial^2 F_N$ which is F_N-invariant

and σ-invariant. The set of all geodesic currents on F_N is denoted by $Curr(F_N)$. The set $Curr(F_N)$ is endowed with the weak-* topology. This makes $Curr(F_N)$ into a locally compact space.

Specifically, let $\alpha : F_N \to \pi_1(\Gamma, x)$ be a simplicial chart on F_N. Let $\mu_n, \mu \in Curr(F_N)$. It is not hard to show [34] that $\lim_{n\to\infty} \mu_n = \mu$ in $Curr(F_N)$ if and only if for every non-trivial reduced edge-path γ in $\widetilde{\Gamma}$ we have

$$\lim_{n\to\infty} \mu_n(Cyl_{\widetilde{\Gamma}}(\gamma)) = \mu(Cyl_{\widetilde{\Gamma}}(\gamma)).$$

Let $\mu \in Curr(F_N)$ and let v be a non-trivial reduced edge-path in Γ. Denote

$$\langle v, \mu \rangle_\alpha := \mu(Cyl_{\widetilde{\Gamma}}(\gamma)),$$

where γ is any edge-path in $\widetilde{\Gamma}$ that is labelled by v. Since μ is F_N-invariant, this definition does not depend on the choice of the lift γ of v.

There is a natural continuous left-action of $Aut(F_N)$ on $Curr(F_N)$ by linear transformations. Namely, let $\psi \in Aut(F_N)$. Then ψ is a quasi-isometry of F_N and hence ψ induces a homeomorphism $\partial\psi$ of ∂F_N and hence a homeomorphism $\partial^2\psi : \partial^2 F_N \to \partial^2 F_N$. Then for $\mu \in Curr(F_N)$ and $S \subseteq \partial^2 F_N$ put

$$(\psi\mu)(S) := \mu(\partial^2\psi^{-1}S).$$

It is not hard to check [34] that $\psi\mu$ is indeed a geodesic current. Moreover, the group of inner automorphisms $Inn(F_N)$ is contained in the kernel of the action of $Aut(F_N)$ on $Curr(F_N)$. Therefore this action factors through to a continuous action of $Out(F_N)$ on $Curr(F_N)$.

Notation 5.5.
(1) For any $g \in F_N \smallsetminus \{1\}$ denote by $g^\infty = \lim_{n\to\infty} g^n$ and $g^{-\infty} = \lim_{n\to\infty} g^n$ the two distinct limit points in ∂F_N. Hence one obtains $(g^{-\infty}, g^\infty) \in \partial^2 F_N$.
(2) For any $g \in F_N$ we will denote by $[g]_{F_N}$ or just by $[g]$ the conjugacy class of g in F_N.

Definition 5.6 (Counting and Rational Currents). (1) Let $g \in F_N$ be a non-trivial element that is not a proper power in F_N. Set

$$\eta_g = \sum_{h \in [g]} \left(\delta_{(h^{-\infty}, h^\infty)} + \delta_{(h^\infty, h^{-\infty})} \right),$$

where $\delta_{(h^{-\infty}, h^\infty)}$ denotes as usually the atomic Dirac (or "counting") measure concentrated at the point $(h^{-\infty}, h^\infty)$.

Let $\mathcal{R}(g)$ be the collection of all F_N-translates of $(g^{-\infty}, g^\infty)$ and $(g^\infty, g^{-\infty})$ in $\partial^2 F_N$. This gives

$$\eta_g = \sum_{(x,y) \in \mathcal{R}(g)} \delta_{(x,y)},$$

and hence η_g is F_N-invariant and flip-invariant, that is $\eta_g \in Curr(F_N)$.

(2) Let $g \in F_N \smallsetminus \{1\}$ be arbitrary. Write $g = f^m$ where $m \geq 1$ and $f \in F_N$ is not a proper power, and define:

$$\eta_g := m \cdot \eta_f.$$

We call $\eta_g \in Curr(F_N)$ the *counting current* given by g. Non-negative scalar multiples of counting currents are called *rational currents*.

It is easy to see that if $[g] = [h]$ then $\eta_g = \eta_h$ and $\eta_g = \eta_{g^{-1}}$. Thus η_g depends only on the conjugacy class of g and we will sometimes denote $\eta_{[g]} := \eta_g$. Moreover, it is not hard to check [34] that for $\varphi \in Out(F_N)$ and $g \in F_N \smallsetminus \{1\}$ we have $\varphi \eta_{[g]} = \eta_{\varphi[g]}$. One can also give a more explicit combinatorial description of the counting current η_g in terms of counting the numbers of occurrences of freely reduced words in a "cyclic word" w representing g (with respect to some fixed free basis of F_N). We refer the reader to [34] for details.

Proposition 5.7. [33, 34] *The set of all rational currents is dense in the space $Curr(F_N)$.*

Definition 5.8 (Projectivized space of geodesic currents)**.** Let $N \geq 2$. We define

$$\mathbb{P}Curr(F_N) = Curr(F_N) \smallsetminus \{0\} / \sim$$

where two currents $\mu_1, \mu_2 \in Curr(F_N) \smallsetminus \{0\}$ satisfy $\mu_1 \sim \mu_2$ if there is some constant $c > 0$ such that $\mu_2 = c \mu_1$. For a nonzero current $\mu \in Curr(F_N)$ denote by $[\mu]$ the projective class of μ, that is, the $\sim$-equivalence class of μ.

The quotient set $\mathbb{P}Curr(F_N)$ inherits the quotient topology from $Curr(F_N)$ as well as a continuous action of $Out(F_N)$. The space $\mathbb{P}Curr(F_N)$ is called the *projectivized space of geodesic currents* (or simply *space of projectivized currents*) on F_N.

It is known [33, 34] that $\mathbb{P}Curr(F_N)$ is compact.

6. THE INTERSECTION FORM

In this section we will prove the main result of this paper, whose slightly simplified version was stated in the Introduction as Theorem A. We state our result now in its full strength, using the terminology introduced in the previous sections.

6.1. **Statement of the main result.**

Theorem 6.1. *Let $N \geq 2$ be an integer. There exists a unique geometric intersection form*

$$\langle \ , \ \rangle : \overline{cv}(F_N) \times Curr(F_N) \to \mathbb{R}_{\geq 0}$$

with the following properties.

(1) *The function $\langle \ , \ \rangle$ is continuous.*

(2) *The function $\langle\ ,\ \rangle$ is $R_{\geq 0}$-homogeneous in the first argument. Namely, for any $T \in \overline{cv}(F_N)$, $\mu \in Curr(F_N)$ and $\lambda \geq 0$ we have:*

$$\langle \lambda T, \mu \rangle = \lambda \langle T, \mu \rangle$$

(3) *The function $\langle\ ,\ \rangle$ is $R_{\geq 0}$-linear in the second argument. Namely, for any $T \in \overline{cv}(F_N)$, $\mu_1, \mu_2 \in Curr(F_N)$ $\lambda_1, \lambda_2 \geq 0$ we have:*

$$\langle T, \lambda_1 \mu_1 + \lambda_2 \mu_2 \rangle = \lambda_1 \langle T, \mu_1 \rangle + \lambda_2 \langle T, \mu_2 \rangle$$

(4) *The function $\langle\ ,\ \rangle$ is $Out(F_N)$-invariant: for any $T \in \overline{cv}(F_N)$, $\mu \in Curr(F_N)$ and $\varphi \in Out(F_N)$ we have:*

$$\langle \varphi T, \varphi \mu \rangle = \langle T, \mu \rangle$$

(5) *For any $T \in \overline{cv}(F_N)$ and any $g \in F_N$, with associated counting current $\eta_g \in Curr(F_N)$, we have:*

$$\langle T, \eta_g \rangle\ =\ ||g||_T$$

(6) *Let $\alpha : F \to \pi_1(\Gamma, x)$ be a simplicial chart on F and let $L : E\Gamma \to \mathbb{R}_{>0}$ be a metric graph structure on Γ and let $T \in cv(F)$ be the tree corresponding to the pull-back of L to $\widetilde{\Gamma}$, with the action of F_N on T via α. Then for any $\mu \in Curr(F_N)$ we have:*

$$\langle \widetilde{\Gamma}, \mu \rangle = \sum_{e \in E^+\Gamma} L(e) \langle e, \mu \rangle_\alpha,$$

where $E\Gamma = E^+\Gamma \sqcup E^-\Gamma$ is an orientation on Γ.

Remark 6.2.

(a) Note that conditions (1), (3) and (5) already imply that if such an intersection form exists, then it is unique. Indeed, recall that the set of rational currents is dense in $Curr(F)$. Thus if $\mu \in Curr(F)$ then there exists a sequence of rational currents $\lambda_i \eta_{g_i}$ such that $\mu = \lim_{i \to \infty} \lambda_i \eta_{g_i}$. Hence the continuity and linearity of the intersection form imply that

$$\langle T, \mu \rangle = \lim_{i \to \infty} \lambda_i ||g_i||_T.$$

Thus Theorem 6.1 implicitly implies that the above limit exists and does not depend on the choice of the sequence of rational currents converging to μ.

(b) For the case of $cv(F_N)$ the statement of Theorem 6.1 was already obtained in [34, 46], where we constructed the intersection form with the required properties on $cv(F_N) \times Curr(F_N)$. The main difficulty that had to be overcome in the present paper is to prove that that intersection form admits a continuous "boundary" extension to a continuous map $\overline{cv}(F_N) \times Curr(F_N) \to \mathbb{R}$.

(c) Note that the $Out(F_N)$-equivariance equality given in part (4) of Theorem 6.1 is equivalent to the formula

$$\langle T\varphi, \mu \rangle = \langle T, \varphi\mu \rangle,$$

as follows directly from the fact that the left side of this equation is equal to $\langle \varphi^{-1}T, \mu \rangle$ (see the definition of the $Out(F_N)$-action in Section 2).

6.2. The case of $cv(F_N)$.

In [34, 46] we established the statement of Theorem 6.1 for $cv(F_N)$:

Proposition-Definition 6.3 (Intersection Form on $cv(F_N)$). Let $N \geq 2$. There exists a unique map

$$\langle \, , \, \rangle : cv(F_N) \times Curr(F_N) \to \mathbb{R}_{\geq 0}$$

satisfying conditions (1)-(6) of Theorem 6.1 for arbitrary $T \in cv(F)$.

For $T \in cv(F_N)$ and $\mu \in Curr(F_N)$ we call $\langle T, \mu \rangle$ the *geometric intersection number of T and μ*.

Note, that, as we have seen in Remark 6.2, if $T \in cv(F_N)$ and $\mu \in Curr(F_N)$ is represented as $\mu = \lim_{i \to \infty} \lambda_i \eta_{g_i}$ for some $g_i \in F_N$ and $\lambda_i \geq 0$ then

$$\langle T, \mu \rangle = \lim_{i \to \infty} \lambda_i ||g_i||_T.$$

6.3. Continuous extension of the intersection form to $\overline{cv}(F_N)$.

The main tool to prove the existence of a continuous extension of the intersection form to $\overline{cv}(F_N)$ will be Proposition 4.1. We first prove:

Proposition 6.4. *Let $T \in \overline{cv}(F_N)$ and let $\mu \in Curr(F_N)$ be such that $\mu = \lim_{i \to \infty} \lambda_i \eta_{g_i}$ for some $g_i \in F_N$ and $\lambda_i \geq 0$. Then the limit*

$$\lim_{i \to \infty} \lambda_i ||g_i||_T$$

exists and does not depend on the choice of the sequence $\lambda_i \eta_{g_i}$ of the rational currents that converges to μ.

Proof. Fix a free basis A of F_N. Let $g_i \in F_N$ and $\lambda_i \geq 0$ be such that $\mu = \lim_{i \to \infty} \lambda_i \eta_{g_i}$. We claim that $\lambda_i ||g_i||_T$ is a Cauchy sequence of real numbers and hence has a finite limit.

Let $\epsilon > 0$ be arbitrary. Choose $0 < \epsilon_1 < \epsilon$ such that $2\epsilon_1(\langle T_A, \mu \rangle + \epsilon_1) + \epsilon_1 \leq \epsilon$. Note that we allow for the possibility that $\mu = 0$.

Let U_{ϵ_1} be the neighborhood of T provided by Proposition 4.1. Choose a tree $T' \in U_{\epsilon_1}$ such that $T' \in cv(F_N)$. Then $\left| ||g_i||_T - ||g_i||_{T'} \right| \leq \epsilon_1 ||g_i||_A$ and hence

$$\left| \lambda_i ||g_i||_T - \lambda_i ||g_i||_{T'} \right| \leq \epsilon_1 \lambda_i ||g_i||_A.$$

Recall that $\lim_{i \to \infty} \lambda_i ||g_i||_{T'} = \langle T', \mu \rangle$ and $\lim_{i \to \infty} \lambda_i ||g_i||_A = \langle T_A, \mu \rangle$ since $T', T_A \in cv(F_N)$.

Thus there is $i_0 \geq 1$ such that for every $i \geq i_0$ we have $|\lambda_i ||g_i||_{T'} - \langle T', \mu \rangle| \leq \epsilon_1$ and $\lambda_i ||g_i||_A \leq \langle T_A, \mu \rangle + \epsilon_1$.

Thus for every $i \geq i_0$ we have

$$\left| \lambda_i ||g_i||_T - \langle T', \mu \rangle \right| \leq \epsilon_1(\langle T_A, \mu \rangle + \epsilon_1) + \epsilon_1.$$

This implies that the sequence $\lambda_i ||g_i||_T$ is bounded and, moreover, for any $i, j \geq i_0$

$$\left| \lambda_i ||g_i||_T - \lambda_j ||g_j||_T \right| \leq 2(\epsilon_1(\langle T_A, \mu \rangle + \epsilon_1) + \epsilon_1) \leq \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this shows that $\lambda_i ||g_i||_T$ is a Cauchy sequence of real numbers and hence has a finite limit in $\mathbb{R}$.

It is now clear that this limit does not depend on the choice of a sequence of rational currents $\lambda_i \eta_{g_i}$ such that $\lim_{i \to \infty} \lambda_i \eta_{g_i} = \mu$, since one can mix any two such sequences together to produce a new sequence of rational currents also limiting to μ.

$\square$

Proposition 6.4 implies that the following notion is well-defined:

Definition 6.5 (Intersection form on $\overline{cv}(F_N)$)**.** Let $T \in \overline{cv}(F_N)$ and let $\mu \in Curr(F_N)$. Put

$$\langle T, \mu \rangle = \lim_{i \to \infty} \lambda_i ||g_i||_T$$

where $g_i \in F_N$ and $\lambda_i \geq 0$ are any such that $\mu = \lim_{i \to \infty} \lambda_i \eta_{g_i}$.

Note that the intersection number from Definition 6.5 agrees with the intersection number from from Proposition-Definition 6.3 for arbitrary $T \in cv(F_N)$ and $\mu \in Curr(F_N)$.

Lemma 6.6. *Let A be a free basis of F_N. Let $T \in \overline{cv}(F_N)$. Let $\epsilon > 0$ and let U_ϵ be the neighborhood of T in $\overline{cv}(F_N)$ provided by Proposition 4.1. Then for any $T_1, T_2 \in U_\epsilon$ and for any $\nu \in Curr(F_N)$ have*

$$\left| \langle T_1, \nu \rangle - \langle T_2, \nu \rangle \right| \leq 2\epsilon \langle T_A, \nu \rangle.$$

Proof. The statement is obvious if $\nu = 0$ so we will assume that $\nu \neq 0$. Hence $\langle T_A, \nu \rangle > 0$ and $\langle T_0, \nu \rangle > 0$. Let $\epsilon_1 > 0$ be such that $\epsilon(\langle T_A, \nu \rangle + \epsilon_1) + 2\epsilon_1 \leq 2\epsilon \langle T_A, \nu \rangle$.

Let $\nu = \lim_{i \to \infty} \lambda_i \eta_{g_i}$ for some $g_i \in F_N$ and $\lambda_i \geq 0$. Choose $i_0 \geq 1$ such that for every $i \geq i_0$

$$\left| \langle T_j, \nu \rangle - \lambda_i ||g_i||_{T_j} \right| \leq \epsilon_1, \text{ for } j = 1, 2$$

and

$$\left| \langle T_A, \nu \rangle - \lambda_i ||g_i||_A \right| \leq \epsilon_1.$$

Then for $i \geq i_0$ we have, by Proposition 4.1:

$$\left| \langle T_1, \nu \rangle - \langle T_2, \nu \rangle \right| \leq \left| \lambda_i ||g_i||_{T_1} - \lambda_i ||g_i||_{T_2} \right| + 2\epsilon_1 \leq$$
$$\leq \epsilon \lambda_i ||g_i||_A + 2\epsilon_1 \leq \epsilon(\langle T_A, \nu \rangle + \epsilon_1) + 2\epsilon_1 \leq 2\epsilon \langle T_A, \nu \rangle.$$

$\square$

Proof of Theorem 6.1. We first show that the map $\langle \, , \, \rangle : \overline{cv}(F_N) \times Curr(F_N) \to \mathbb{R}_{\geq 0}$, given in Definition 6.5, is continuous.

Choose a free basis A of F_N, and let $T \in \overline{cv}(F_N)$, $\mu \in Curr(F_N)$ and $\epsilon > 0$ be arbitrary.

Let $\epsilon_1 > 0$ be such that $4\epsilon_1 \langle T_A, \mu \rangle \leq \epsilon/2$. Let $\epsilon_2 > 0$ be such that $2\epsilon_1\epsilon_2 + \epsilon_2 \leq \epsilon/2$.

Let $U_{\epsilon_1} \subseteq \overline{cv}(F_N)$ be the neighborhood of T in $\overline{cv}(F_N)$ provided by Proposition 4.1. Choose $T_0 \in U_{\epsilon_1} \cap cv(F_N)$.

Since $\langle \ , \ \rangle : cv(F_N) \times Curr(F_N) \to \mathbb{R}$ is continuous and since $T_0, T_A \in cv(F_N)$, there exists a neighborhood V of μ in $Curr(F_N)$ such that for every $\mu' \in V$ we have

$$\left| \langle T_0, \mu' \rangle - \langle T_0, \mu \rangle \right| \leq \epsilon_2$$

and

$$\left| \langle T_A, \mu' \rangle - \langle T_A, \mu \rangle \right| \leq \epsilon_2.$$

Now let $T' \in U_{\epsilon_1}$ and $\mu' \in V$ be arbitrary. By Lemma 6.6 we have

$$\left| \langle T', \mu' \rangle - \langle T, \mu \rangle \right| \leq$$
$$\left| \langle T', \mu' \rangle - \langle T_0, \mu' \rangle \right| + \left| \langle T_0, \mu' \rangle - \langle T_0, \mu \rangle \right| + \left| \langle T_0, \mu \rangle - \langle T, \mu \rangle \right| \leq$$
$$\leq 2\epsilon_1 \langle T_A, \mu' \rangle + \epsilon_2 + 2\epsilon_1 \langle T_A, \mu \rangle \leq$$
$$2\epsilon_1 \langle T_A, \mu \rangle + 2\epsilon_1\epsilon_2 + \epsilon_2 + 2\epsilon_1 \langle T_A, \mu \rangle = 4\epsilon_1 \langle T_A, \mu \rangle + 2\epsilon_1\epsilon_2 + \epsilon_2 \leq \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this implies that $\langle \ , \ \rangle : \overline{cv}(F_N) \times Curr(F_N) \to \mathbb{R}$ is continuous at the point (T, μ). Since $(T, \mu) \in \overline{cv}(F_N) \times Curr(F_N)$ was arbitrary, it follows that $\langle \ , \ \rangle : \overline{cv}(F_N) \times Curr(F_N) \to \mathbb{R}$ is continuous, as required. This establishes part (1) of Theorem 6.1.

The fact that parts (1)-(5) of Theorem 6.1 hold now follows by continuity from the same properties known to hold for $\langle \ , \ \rangle : cv(F_N) \times Curr(F_N) \to \mathbb{R}$. Part (6) of Theorem 6.1 only concerns $\mathbb{R}$-trees from $cv(F_N)$ and is thus already known (see Proposition-Definition 6.3 above). $\qquad\square$

7. THE INTERSECTION FORM AND IWIP AUTOMORPHISMS OF F_N

Notation 7.1. Note that if $T, T' \in \overline{cv}(F_N)$ and $\mu, \mu' \in Curr(F_N), \mu \neq 0, \mu' \neq 0$ are such that $[T] = [T']$ and $[\mu] = [\mu']$ then $\langle T, \mu \rangle = 0$ if and only if $\langle T', \mu' \rangle = 0$. Therefore for $x \in \overline{CV}(F_N)$, $y \in \mathbb{P}Curr(F_N)$ we will write $\langle x, y \rangle = 0$ if for some (or equivalently, for any) $T \in \overline{cv}(F_N)$, $\mu \in Curr(F_N)$ with $[T] = x$ and $[\mu] = y$ we have $\langle T, \mu \rangle = 0$.

Lemma 7.2. *Let* $[T_n], [T] \in \overline{CV}(F_N)$ *and* $[\mu_n] \in \mathbb{P}Curr(F_N)$ *be such that* $\lim_{n\to\infty} [T_n] = [T]$ *and* $\lim_{n\to\infty} [\mu_n] = [\mu]$, *and such that* $\langle [T_n], [\mu_n] \rangle = 0$ *for every* $n \geq 1$. *Then*

$$\langle [T], [\mu] \rangle = 0.$$

Proof. There exist $r_n \geq 0$ and $c_n \geq 0$ such that $T = \lim_{n\to\infty} r_n T_n$ and $\mu = \lim_{n\to\infty} c_n \mu_n$. By linearity of the intersection form we have $\langle r_n T_n, c_n \mu_n \rangle = r_n c_n \langle T_n, \mu_n \rangle = 0$. Hence by continuity (part (1) of Theorem 6.1) we have $\langle T, \mu \rangle = \lim_{n\to\infty} \langle r_n T_n, c_n \mu_n \rangle = 0$, as required. $\qquad\square$

Definition 7.3 (IWIP). As in [45], we say that an outer automorphism $\varphi \in Out(F_N)$ is *irreducible with irreducible powers* or an *iwip* if no conjugacy class of any non-trivial proper free factor of F_N is mapped by a positive power of φ to itself.

It is known that if such an iwip φ is without periodic conjugacy classes, then φ has a "North-South" dynamics for its induced actions on both, $\overline{CV}(F_N)$ and $\mathbb{P}Curr(F_N)$:

Proposition 7.4. *Let $N \geq 3$ and let $\varphi \in Out(F_N)$ be irreducible with irreducible powers. Then the following hold:*

(1) *(Levitt-Lustig [45]) The action of φ on $\overline{CV}(F_N)$ has precisely two distinct fixed points, $[T_+]$ and $[T_-]$, that both belong to $\partial CV(F_N)$. Moreover, for any $[T] \neq [T_-]$ in $\overline{CV}(F_N)$ we have $\lim_{n\to\infty} \varphi^n[T] = [T_+]$. Similarly, for any $[T] \neq [T_+]$ in $\overline{CV}(F_N)$ we have $\lim_{n\to\infty} \varphi^{-n}[T] = [T_-]$.*

(2) *(Reiner Martin [48]) Suppose in addition that φ has no periodic conjugacy classes in F_N. Then the action of φ on $\mathbb{P}Curr(F_N)$ has precisely two distinct fixed points $[\mu_+]$ and $[\mu_-]$. Moreover, for any $[\mu] \neq [\mu_-]$ in $\mathbb{P}Curr(F_N)$ we have $\lim_{n\to\infty} \varphi^n[\mu] = [\mu_+]$. Similarly, for any $[\mu] \neq [\mu_+]$ in $\mathbb{P}Curr(F_N)$ we have $\lim_{n\to\infty} \varphi^{-n}[\mu] = [\mu_-]$.*

Convention 7.5. For the remainder of this section, unless specified otherwise, let $N \geq 3$ and let $\varphi \in Out(F_N)$ be irreducible with irreducible powers, and without periodic conjugacy classes. Let $[T_+], [T_-] \in \partial CV(F_N)$ be the attracting and repelling fixed points for the (left) action of φ on $\overline{CV}(F_N)$. Similarly, let $[\mu_+], [\mu_-] \in \mathbb{P}Curr(F_N)$ be the attracting and repelling fixed points for the action of φ on $\mathbb{P}Curr(F_N)$.

Remark 7.6. (1) We would like to alert the reader that the *forward limit tree* of φ, denoted in [18] and [47] by T_φ, is the tree T_- (and not T_+). This is due to the fact that in this paper φ acts on $\mathbb{R}$-trees in $\overline{CV}(F_n)$ from the left, while [18] and [47] in one considers the right-action (compare the discussion in Section 2).

(2) Some useful information about iwips and their induced action on Outer space has been worked out in detail in [47], §4 and §5. A summary of the most important facts is given in [18], Remark 5.5.

(3) An alternative proof (relying on the main result of [38]) for Proposition 7.7 below is given by Proposition 5.6 of [18].

Proposition 7.7. *Let $\varphi, T_\pm$ and $\mu_\pm$ be as in Convention 7.5. Then*

$$\langle T_-, \mu_+ \rangle \neq 0 \quad and \quad \langle T_+, \mu_- \rangle \neq 0.$$

Proof. Let $\alpha : F_N \to \pi_1(\Gamma)$ be a marked graph structure on F_N, given by a train track map on Γ that represents φ, with a metric structure L on the edges of Γ given by the Perron-Frobenius eigen-vector of the transition

matrix (see [4] or Section 3 of [47] for a detailed exposition). Let $T = \widetilde{\Gamma} \in cv(F_N)$ be the discrete $\mathbb{R}$-tree given by the universal covering of Γ, provided with the metric d_L given by the lift of L, and with the action of F_N coming from the marking α.

Let $\lambda > 1$ be the train-track stretching constant for Γ (i.e. the Perron-Frobenius eigen-value of the transition matrix of the train track map). It is known (see, for example, Remark 5.4 of [47]) that $\lim_{n\to\infty} \frac{1}{\lambda^n} \varphi^{-n} T = T_-$.

Let $g \in F_N$, $g \neq 1$ be arbitrary. Then there exists constants $C > 1$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have

$$\frac{1}{C}\lambda^n \leq ||\varphi^n(g)||_T \leq C\lambda^n.$$

The upper bound is derived in Section 3 of [47] before Remark 3.4: The inequality becomes an equality if g is represented by a legal loop. The lower bound follows from the fact that every path in Γ has an iterate (under the train track map) that is a legal composition of legal paths and INP's, see [4].

Note that $||\varphi^n(g)||_T = ||g||_{\varphi^{-n}T}$. It was proved by Reiner Martin [48] that $\lim_{n\to\infty}[\varphi^n \eta_g] = [\mu_+]$ and, moreover, that, after possibly multiplying μ_+ by a positive scalar, we have $\lim_{n\to\infty} \frac{1}{\lambda^n} \eta_{\varphi^n(g)} = \mu_+$. We compute:

$$\langle T_-, \mu_+ \rangle = \lim_{n\to\infty} \langle \frac{1}{\lambda^n}\varphi^{-n}T, \frac{1}{\lambda^n}\eta_{\varphi^n(g)} \rangle$$

$$= \lim_{n\to\infty} \frac{1}{\lambda^{2n}}\langle \varphi^{-n}T, \eta_{\varphi^n(g)} \rangle = \lim_{n\to\infty} \frac{1}{\lambda^{2n}}\langle T, \varphi^n \eta_{\varphi^n(g)} \rangle$$

$$= \lim_{n\to\infty} \frac{1}{\lambda^{2n}}\langle T, \eta_{\varphi^{2n}(g)} \rangle = \lim_{n\to\infty} \frac{1}{\lambda^{2n}}||\varphi^{2n}(g)||_T$$

$$\geq \lim_{n\to\infty} \frac{1}{\lambda^{2n}}C\lambda^{2n} = C > 0$$

Replacing φ by φ^{-1} we conclude that $\langle T_+, \mu_- \rangle > 0$ as well.

$\square$

Proposition 7.8. *Let $[T_n] \in \overline{CV}(F_N)$ and $[\mu_n] \in \mathbb{P}Curr(F_N)$ be sequences such that*

$$\langle [T_n], [\mu_n] \rangle = 0$$

for every $n \geq 1$. Then we have:

$$\lim_{n\to\infty}[T_n] = [T_+] \iff \lim_{n\to\infty}[\mu_n] = [\mu_+].$$

Proof. Let $\lim_{n\to\infty}[T_n] = [T_+]$. Suppose that $\lim_{n\to\infty}[\mu_n] \neq [\mu_+]$.

Since $\mathbb{P}Curr(F_N)$ is compact, after passing to a subsequence we may assume that $\lim_{n\to\infty}[\mu_n] = [\mu]$ for some $[\mu] \neq [\mu_+]$ in $\mathbb{P}Curr(F_N)$. Note that by Lemma 7.2 we have $\langle [T_+], [\mu] \rangle = 0$. Since $[\mu] \neq [\mu_+]$, part (2) of Proposition 7.4 implies that $\lim_{n\to\infty} \varphi^{-n}[\mu] = [\mu_-]$. Note that $[T_+]$ is fixed by φ^{-1}, and

that we have

$$\langle [T_+], \varphi^{-n}[\mu] \rangle = \langle \varphi^{-n}[T_+], \varphi^{-n}[\mu] \rangle = \langle [T_+], [\mu] \rangle = 0$$

for every $n \geq 1$. Hence Lemma 7.2 implies $\langle [T_+], [\mu_-] \rangle = 0$. This is to say that $\langle T_+, \mu_- \rangle = 0$, yielding a contradiction with Proposition 7.7. Thus $\lim_{n\to\infty} [\mu_n] = [\mu_+]$.

The argument for the other direction is completely symmetric. $\qquad\square$

8. Curve complex analogues for free groups

8.1. The bipartite intersection graph.

Definition 8.1 (Intersection graph). Let $\mathcal{I} = \mathcal{I}(F_N)$ be a bipartite graph defined as follows. The vertex set of $\mathcal{I}$ is $V\mathcal{I} = \overline{CV}(F_N) \cup \mathbb{P}Curr(F_N)$. Two vertices $[T] \in \overline{CV}(F_N)$ and $[\mu] \in \mathbb{P}Curr(F_N)$ are connected by an edge in $\mathcal{I}$ if and only if $\langle [T], [\mu] \rangle = 0$.

Since the intersection form is $Out(F_N)$-invariant, the graph $\mathcal{I}(F_N)$ comes equipped with a natural action of $Out(F_N)$ by graph automorphisms.

It is not hard to show that for $N \geq 3$ there is a single connected component $\mathcal{I}_0(F_N)$ of $\mathcal{I}(F_N)$ containing all $[T]$ for the Bass-Serre trees T corresponding to all nontrivial free product decompositions of F_N. Moreover, the same connected component $\mathcal{I}_0(F_N)$ also contains η_a for all nontrivial $a \in F_N$ that belong to some proper free factors of F_N. It is also not hard to show that $\mathcal{I}_0(F_N)$ is $Out(F_N)$-invariant.

Note also that there are many connected components in this graph. Indeed, every vertex $[T] \in CV(F_N)$ is an isolated point, and it follows from [18] that many pairs $([T], [\mu])$ form a single edge connected component, in particular all pairs $([T_+], [\mu_+])$ as in Convention 7.5.

Proposition 8.2. *Let $[T_n], [T] \in \overline{CV}(F_N)$ be such that $[T] \neq [T_+]$ and that $\lim_{n\to\infty} [T_n] = [T_+]$, for $[T_+]$ as in Convention 7.5. Then in the graph $\mathcal{I}$ we have:*

$$\lim_{n\to\infty} d_{\mathcal{I}}([T_n], [T]) = \infty.$$

Proof. Suppose that the statement of the lemma fails. Then there exists a sequence $[T_n] \in \overline{CV}(F_N)$ with $\lim_{n\to\infty} [T_n] = [T_+]$, such that $\max_{n\geq 1} d_{\mathcal{I}}([T_n], [T]) < \infty$. Among all sequences $[T_n] \in \overline{CV}(F_N)$ satisfying $\lim_{n\to\infty} [T_n] = [T_+]$ and $\max_{n\geq 1} d_{\mathcal{I}}([T_n], [T]) < \infty$, choose a sequence $[T_n]$ such that $\max_{n\geq 1} d_{\mathcal{I}}([T_n], [T])$ is the smallest possible.

Let $D = \max_{n\geq 1} d_{\mathcal{I}}([T_n], [T])$. Suppose that $D > 0$. Then, after passing to a further subsequence, we may assume that $[T_n] \neq [T]$ for every $n \geq 1$. Note that by definition of the graph $\mathcal{I}$, the numbers D and $d_{\mathcal{I}}([T_n], [T])$ are positive even integers. By definition of $\mathcal{I}$ it follows that there exist $[T_n'] \in \overline{CV}(F_N)$ such that $d_{\mathcal{I}}([T_n], [T_n']) = 2$ and $d_{\mathcal{I}}([T_n'], [T]) = d_{\mathcal{I}}([T_n], [T]) - 2$.

Hence, again by definition of $\mathcal{I}$, there exists a sequence $[\mu_n] \in \mathbb{P}Curr(F_N)$ such that $\langle [T_n], [\mu_n] \rangle = 0 = \langle [T_n'], [\mu_n] \rangle$. Since $\lim_{n\to\infty} [T_n] = [T_+]$ and $\langle [T_n], [\mu_n] \rangle = 0$, Proposition 7.8 implies that $\lim_{n\to\infty} [\mu_n] = [\mu_+]$. Since $\langle [T_n'], [\mu_n] \rangle = 0$, Proposition 7.8 then implies that $\lim_{n\to\infty} [T_n'] = [T_+]$. Thus $\lim_{n\to\infty} [T_n'] = [T_+]$ and $\max_{n \geq 1} d_{\mathcal{I}}([T_n'], [T]) = D - 2 < D = \max_{n \geq 1} d_{\mathcal{I}}([T_n'], [T])$. This contradicts the minimality in the choice of $[T_n]$. Therefore we conclude $D = 0$. Thus $0 = \max_{n \geq 1} d_{\mathcal{I}}([T_n], [T])$ and hence $[T_n] = [T]$ for every $n \geq 1$. This contradicts the assumptions that $[T] \neq [T_+]$ and that $\lim_{n\to\infty} [T_n] = [T_+]$. $\qquad \square$

Proposition 8.2 and Proposition 7.4 immediately imply:

Corollary 8.3. *Let* $\varphi, [T_+]$ *and* $[T_-]$ *be as in Convention 7.5, and let* $[T] \in \overline{CV}(F_N)$ *be such that* $[T] \neq [T_+], [T_-]$. *Then in the intersection graph* $\mathcal{I} = \mathcal{I}(F_N)$ *we have:*

$$\lim_{n\to\infty} d_{\mathcal{I}}(\varphi^n[T], [T]) = \infty$$

8.2. **Other curve complex analogues.**

One can define several other natural free group analogues of the curve complex. Each of them will admit an $Out(F_N)$-equivariant Lipschitz map into the intersection graph

Definition 8.4. Let $N \geq 3$.
  (1) The *free splitting graph* $\mathcal{F} = \mathcal{F}(F_N)$ is a simple graph whose vertices are non-trivial splitting of F_N as the fundamental group of a graph-of-groups with a single non-loop edge and trivial edge group (so algebraically they correspond to nontrivial free product decompositions of F_N). Two such splittings are considered equal if their Bass-Serre trees are F_N-equivariantly isometric, that is, if they equal as points of $cv(F_N)$. Two distinct splittings $T_1, T_2 \in V\mathcal{F}(F_N)$ are adjacent in $\mathcal{F}$ if there exists a splitting of F_N as the fundamental group of a graph-of-groups with two (non-loop) edges and trivial edge groups, such that T_1 is obtained by collapsing one edge of this graph-of-groups, and T_2 is obtained by collapsing the other edge.
  (2) The *cut graph* $\mathcal{S} = \mathcal{S}(F_N)$ is a simple graph whose vertices are non-trivial splitting of F_N as the fundamental group of a graph-of-groups with a single edge (possible a loop edge) and trivial edge group. Again, two such splittings are considered equal if their Bass-Serre trees are F_N-equivariantly isometric, that is, if they equal as points of $cv(F_N)$. Two distinct splittings $T_1, T_2 \in V\mathcal{F}(F_N)$ are adjacent in $\mathcal{F}$ if there exists a splitting of F_N as the fundamental group of a graph-of-groups with two (non-loop) edges and trivial edge groups, such that T_1 is obtained by collapsing one edge of this graph-of-groups, and T_2 is obtained by collapsing the other edge.

(3) The *dual free splitting graph* $\mathcal{F}^* = \mathcal{F}^*(F_N)$ has the same vertex set as $\mathcal{F}(F_N)$. Two distinct vertices of $\mathcal{F}^*(F_N)$ corresponding to splittings T_1 and T_2 are adjacent whenever there exists a nontrivial element $g \in F_N$ such that $||g||_{T_1} = ||g||_{T_2} = 0$. One can show that this adjacency condition is equivalent to saying that there exists a nontrivial *primitive* (i.e. a member of a free basis of F_N) element a of F_N such that $||a||_{T_1} = ||a||_{T_2} = 0$.

(4) The *dual cut graph* $\mathcal{S}^* = \mathcal{S}^*(F_N)$ has the same vertex set as $\mathcal{S}(F_N)$. Two distince vertices are adjacent in $\mathcal{S}^*(F_N)$ whenever the corresponding splittings of F_N have some common nontrivial elliptic element.

(5) The *ellipticity graph* $\mathcal{Z} = \mathcal{Z}(F_N)$ is a bipartite graph. Its vertex set is the disjoint union of the vertex set of $\mathcal{S}(F_N)$ and the set of all F_N-conjugacy classes $[a]$ of nontrivial elements $a \in F_N$. A vertex $[a]$ is adjacent to a vertex T whenever a is an elliptic element with respect to T, that is $||a||_T = 0$ (algebraically, this means that a is conjugate to a vertex group element for the free product splitting T).

(6) The *free factor graph* $\mathcal{J} = \mathcal{J}(F_N)$ is a simple graph defined as follows: The vertex set of of $\mathcal{J}$ is the set of conjugacy classes in F_N of all free factors A of F_N such that $A \neq 1, A \neq F_N$. Two distinct vertices $x, y \in V\mathcal{J}$ are adjacent in $\mathcal{J}$ if and only if for some A, B with $[A] = x$ and $[y] = B$ there exists $C \leq F_N$ such that $F_N = A * B * C$. Note that we allow the case where $C = 1$.

(7) The *dominance graph* $\mathcal{D} = \mathcal{D}(F_N)$ is defined as follows. Put $V\mathcal{D} = V\mathcal{J}$. For distinct $x, y \in V\mathcal{D}$ we say that x, y are adjacent in $\mathcal{D}$ if and only if there exist $A, B \leq F_N$ such that $x = [A]$, $y = [B]$ and such that either $A \leq B$ or $B \leq A$. The dominance graph is precisely the one-skeleton of the "complex of free factors" CF_N whose homotopy properties have been studied by Hatcher and Vogtmann [28].

(8) The *primitivity graph* $\mathcal{P}(F_N)$ whose vertices are conjugacy classes of primitive elements of F_N and where two such conjugacy classes are adjacent whenever there exist a free basis X of F_N and some representatives a_1, a_2 of these conjugacy classes such that $a_1, a_2 \in X$.

It is not hard to see that for $N \geq 3$ all of these graphs are connected and they come equipped with natural $Out(F_N)$-actions. Moreover, with a bit of work, one can show that for a given $N \geq 3$ there are at most two substantially distinct objects among the above graphs.

More specifically, the graphs $\mathcal{F}(F_N)$ and $\mathcal{S}(F_N)$ are quasi-isometric (this is almost immediate from the definitions). Moreover, for $N \geq 3$ the graphs $\mathcal{F}^*(F_N)$, $\mathcal{S}^*(F_N)$, $\mathcal{Z}(F_N)$, $\mathcal{J}(F_N)$, $\mathcal{D}(F_N)$ and $\mathcal{P}(F_N)$ are all quasi-isometric. Also, the full subgraph of $\mathcal{Z}(F_N)$ induced by all the vertices coming from $\mathcal{S}(F_N)$ together with conjugacy classes of primitive elements of F_N can be

shown to be a co-bounded subset of $\mathcal{Z}(F_N)$ and is thus quasi-isometric to $\mathcal{Z}(F_N)$.

Note that there are natural $Out(F_N)$-equivariant Lipshitz maps $j : \mathcal{F}(F_N) \to \mathcal{F}^*(F_N)$ and $q : \mathcal{F}^*(F_N) \to \mathcal{I}_0(F_N)$. The map j is the identity map on the vertex set of $\mathcal{F}(F_N)$ (recall that by definition the vertex sets of $\mathcal{F}(F_N)$ and $\mathcal{F}^*(F_N)$ are equal). The map q sends a vertex T of $\mathcal{F}^*(F_N)$ to a vertex $[T]$ of $\mathcal{I}_0(F_N)$.

Note that if T_1 and T_2 are adjacent in $\mathcal{F}(F_N)$ and the tree T corresponds to their common refinement, then for any nontrivial elliptic element a for T we have $||a||_T = ||a||_{T_1} = ||a||_{T_2} = 0$. Hence T_1 and T_2 are adjacent in $\mathcal{F}^*(F_N)$ and therefore the map j is 1-Lipshitz.

Similarly, suppose that two vertices T_1 and T_2 of $\mathcal{F}^*(F_N)$ are adjacent in $\mathcal{F}^*(F_N)$. Hence there exists a nontrivial element $a \in F_N$ such that $||a||_{T_1} = ||a||_{T_2} = 0$. Hence by the properties of the intersection form $\langle T_i, \eta_a \rangle = ||a||_{T_i} = 0$ for $i = 1, 2$. Hence both $[T_1]$ and $[T_2]$ are adjacent to $[\eta_a]$ in $\mathcal{I}(F_N)$ which implies that the map q is 2-Lipshitz.

It appears (although we do not know how to prove this) that the map j, although Lipshitz, is not a quasi-isometry, and that the fibers of j have infinite diameter as subsets of $\mathcal{F}^*(F_N)$.

Since the maps q, j and $q \circ j$ are Lipshitz, Corollary 8.3 immediately implies analogous statements for the above graphs:

Corollary 8.5. *Let $N \geq 3$ and let $\varphi \in Out(F_N)$ be an atoriodal iwip. Let $\mathcal{Y}$ be one of the graphs $\mathcal{F}(F_N)$, $\mathcal{F}^*(F_N)$. Then for any vertex x of $\mathcal{Y}$ we have:*

$$\lim_{n \to \infty} d_{\mathcal{Y}}(\varphi^n x, x) = \infty.$$

In particular, $diam(\mathcal{Y}) = \infty$.

The above corollary, together with Corollary 8.3, implies Theorem B from the introduction.

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Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA
http://www.math.uiuc.edu/~kapovich/
E-mail address: kapovich@math.uiuc.edu

Mathématiques (LATP), Université Paul Cézanne - Aix Marseille III, ave. Escadrille Normandie-Niémen, 13397 Marseille 20, France
E-mail address: Martin.Lustig@univ-cezanne.fr